Math 756 • Midterm Exam • Victor Matveev
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Problem 1

As for any conformal map, it is sufficient to examine the transformation of the boundary of the domain, which in this case is the unit circle:

$$z=\frac{2\zeta}{\zeta-1}\Rightarrow z(e^{i\theta})=2\frac{\exp(i\theta)}{\exp(i\theta)-1}\cdot\frac{\exp(-i\theta)-1}{\exp(-i\theta)-1}=2\frac{1-\exp(i\theta)}{2-2\cos\theta}=\frac{1-\cos\theta-i\sin\theta}{1-\cos\theta}$$

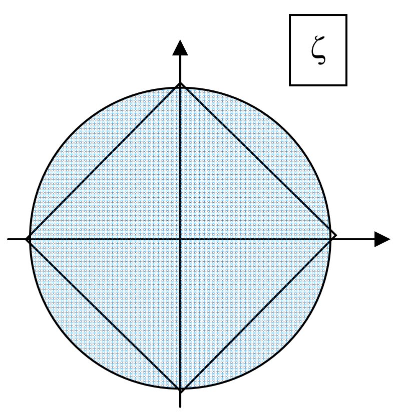
$$\Rightarrow z=1-i\frac{\sin\theta}{1-\cos\theta}$$ Note: we can simplify even further: $z=1-i\dfrac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}=1-i\cot\frac{\theta}{2}$

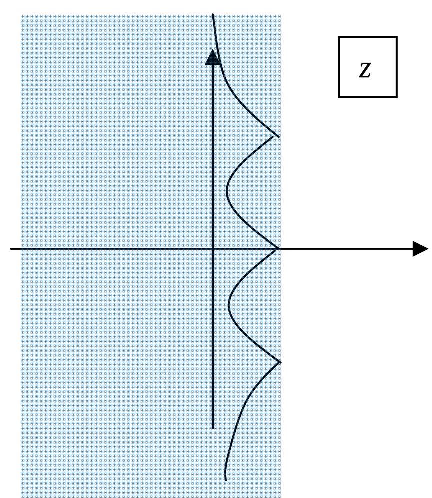
Another method: $\dfrac{2\exp(i\theta)}{\exp(i\theta)-1}=\dfrac{2\exp(i\theta/2)}{\exp(i\theta/2)-\exp(-i\theta/2)}=\dfrac{\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}}{i\sin\frac{\theta}{2}}=1-i\cot\frac{\theta}{2}$

So the image of the unit circle is a vertical line Re(z)=1

Since the center of the disk is transformed into z(0)=0, the disk is transformed into half-plane to the left of the line Re(z)=1.

Transformation of the square: $z(1)=+\infty$, $z(-1)=-2/(-2)=1$; to get $z(\pm i)$ set θ to $\pm\pi/2$ in above formula, or calculate directly: $z(\pm i)=\dfrac{\pm 2i}{\pm i-1}=\left(\dfrac{\pm 2i}{\pm i-1}\right)\left(\dfrac{\mp i-1}{\mp i-1}\right)=\dfrac{2\mp 2i}{2}=1\mp i$

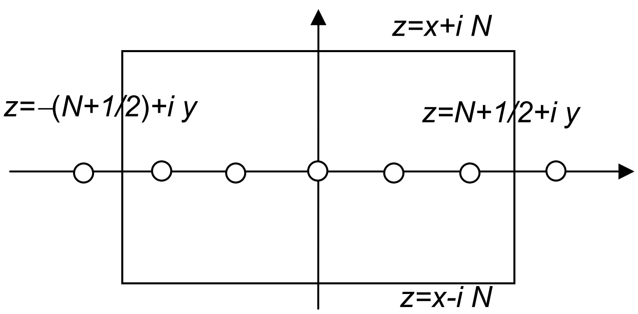




Angles are preserved, so the curved segments in the image of the square meet at right angles

Problem 2

When calculating $\sum_{n=1}^{\infty}\frac{1}{n^k}$ by integrating $f(z)=\frac{\pi\cos\pi z}{z^k\sin\pi z}$, the only purpose of the numerator is to cancel the alternating sign coming from the derivative of $\sin\pi z$ at the pole *z*=*n*. Therefore, we have to get rid of the cosine, and integrate $f(z)=\frac{\pi}{z^2\sin\pi z}$ over the same rectangular contour to find $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$.



Along horizontal segments 1 / |sin(*πz*)| ≤ 1 / |exp(πN)-exp(-πN)|, exponentially decaying as *N* = | Im *z* | →∞, therefore the integral also approaches zero in this limit. Along the vertical segments *z*=±(*N*+*1/2*)+*i y*, we have

$$\frac{1}{|\sin\pi z|}=\frac{1}{\left|\sin\left(\pm\left(N+\tfrac{1}{2}\right)\pi+i\pi y\right)\right|}=\frac{1}{\left|\sin\left(\pm\left(N+\tfrac{1}{2}\right)\pi\right)\cosh\pi y+i\cos\left(\pm\left(N+\tfrac{1}{2}\right)\pi\right)\sinh\pi y\right|}=$$

$$=\frac{1}{\left|\sin\left(\pm\left(N+\tfrac{1}{2}\right)\pi\right)\cosh\pi y\right|}=\frac{1}{\cosh\pi y}\le 1$$

So the integral is bounded by the length of the contour, 2*N*, times 1/|z|^2~1/*N*^{2}, yielding zero in the limit *N*→∞. Thus, the sum of residues is zero:

$$\sum_{n=-\infty}^{\infty}\mathrm{Res}\left(f(z),z_n=n\right)=\mathrm{Res}\left(f(z),0\right)+2\sum_{n=1}^{\infty}\mathrm{Res}\left(f(z),n\right)=\mathrm{Res}\left(f(z),0\right)+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\Rightarrow\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=-\frac{\mathrm{Res}\left(f(z),0\right)}{2}$$

Now let's find the residue at 0:

$$\frac{\pi}{z^2\sin\pi z}=\frac{\pi}{z^2\left(\pi z-\frac{(\pi z)^3}{3!}+O(z^5)\right)}=\frac{1}{z^3}\frac{1}{1-\frac{(\pi z)^2}{6}+O(z^4)}=\frac{1}{z^3}\left(1+\frac{(\pi z)^2}{6}+O(z^4)\right)=\frac{1}{z^3}+\frac{\pi^2}{6}\frac{1}{z}+O(z)$$

$$\Rightarrow\quad\boxed{\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=-\frac{\pi^2}{12}}$$

The answer is negative since the first (dominant) term is negative.

Problem 3

Solve the following Volterra integral equation. You may use without derivation the Laplace transform result $L[\sin\omega t] = \dfrac{\omega}{s^2+\omega^2}$; denote $\gamma = \omega(\omega-1)$ to simplify algebra. Does solution exist when $\omega = +1$?

$$y(t) = t + \int_0^t y(t-\tau)\,\sin\omega\tau\,d\tau$$

$$Y(s) = \frac{1}{s^2} + Y(s)\frac{\omega}{s^2+\omega^2} \;\Rightarrow\; Y(s)\left[1-\frac{\omega}{s^2+\omega^2}\right] = \frac{1}{s^2} \;\Rightarrow\; Y(s)\frac{s^2+\overbrace{\omega^2-\omega}^{\gamma=\omega(\omega-1)}}{s^2+\omega^2} = \frac{1}{s^2} \;\Rightarrow\; Y(s) = \frac{s^2+\omega^2}{s^2\left(s^2+\gamma\right)}$$

Since Y(s) is not that messy, partial fractions can be used:

$$Y(s) = \frac{s^2}{s^2\left(s^2+\gamma\right)} + \frac{\omega^2}{s^2\left(s^2+\gamma\right)} = \frac{1}{s^2+\gamma} + \frac{\omega^2}{\gamma}\left[\frac{1}{s^2} - \frac{1}{s^2+\gamma}\right] = \frac{\omega^2}{\gamma}\frac{1}{s^2} + \underbrace{\left(1-\frac{\omega^2}{\gamma}\right)}_{-\omega/\gamma}\frac{1}{s^2+\gamma} = \boxed{\frac{\omega^2}{\gamma}\frac{1}{s^2} - \frac{\omega}{\gamma}\frac{1}{s^2+\gamma}}$$

Since $L[\sin\omega t] = \dfrac{\omega}{s^2+\omega^2}$, we infer that $L^{-1}\left[\dfrac{1}{s^2+\gamma}\right] = \dfrac{1}{\sqrt{\gamma}}\sin\left(\sqrt{\gamma}t\right)$ therefore:

$$\Rightarrow \quad y(t) = \frac{\omega^2}{\gamma}t - \frac{\omega}{\gamma^{3/2}}\sin\sqrt{\gamma}t = \boxed{\frac{\omega}{\omega-1}t - \frac{1}{\sqrt{\omega(\omega-1)^3}}\sin\sqrt{\omega(\omega-1)}\,t}$$

Singularity ω=1 is removable: γ=0 $\Rightarrow$ $Y(s) = \dfrac{s^2+1}{s^2\left(s^2+0\right)} = \dfrac{1}{s^2} + \dfrac{1}{s^4}$ $\Rightarrow$ $\boxed{Y(s) = t + \dfrac{t^3}{6}}$

(Note: for 0<ω<1 the above results are still correct, yielding $y(t) = \dfrac{\omega}{\omega-1}t + \dfrac{1}{\sqrt{\omega(1-\omega)^3}}\sinh\sqrt{\omega(1-\omega)}\,t$)

Problem 4

Use the Fourier transform to write down the solution for the following modification of the heat (or diffusion) equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} - \omega \dfrac{\partial u}{\partial x}; \quad -\infty < x < +\infty, \quad t > 0, \\ u(x,t=0) = f(x) \end{cases} \Rightarrow \begin{cases} \dfrac{\partial U(k,t)}{\partial t} = \left(-k^2 - ik\omega\right) U(k,t) \\ U(k,0) = F(k) \end{cases}$$

$$\Rightarrow U(k,t) = F(k) \underbrace{\exp\left(-(k^2 + ik\omega)t\right)}_{G(k)} \quad \Rightarrow \quad u(x,t) = \int_{-\infty}^{+\infty} f(x-\xi)\, g(\xi,t)\, d\xi$$

where the Green's function is

$$g(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \underbrace{\exp\left(-(k^2 + ik\omega)t\right)}_{G(k)} \exp(ikx)\, dk = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\left(-k^2 t + ik(x - \omega t)\right) dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\left(-t\left[k^2 + ik\frac{x-\omega t}{t}\right]\right) dk = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\left(-t\left[\left(k + i\frac{x-\omega t}{2t}\right)^2 + \left(\frac{x-\omega t}{2t}\right)^2\right]\right) dk$$

$$= \frac{\exp\left(-\dfrac{(x-\omega t)^2}{4t}\right)}{2\pi} \int_{-\infty}^{+\infty} \exp\left(-t \underbrace{\left(k + i\frac{x-\omega t}{2t}\right)}_{s}^2\right) dk = \frac{\exp\left(-\dfrac{(x-\omega t)^2}{4t}\right)}{2\pi} \underbrace{\int_{-\infty + i\frac{x-\omega t}{2t}}^{+\infty + i\frac{x-\omega t}{2t}} \exp\left(-ts^2\right) ds}_{\sqrt{\frac{\pi}{t}}}$$

$$\Rightarrow \quad \boxed{g(x,t) = \frac{\exp\left(-\dfrac{(x-\omega t)^2}{4t}\right)}{\sqrt{4\pi t}}}$$

Note: same Green's function as in the usual diffusion equation, but with the peak moving with velocity ω

As time evolves, the spread of the Green's function increases, and its center moves to the right

The extra term describes the flow of the medium to the right (transport / advection), with velocity ω.